

Non-Weighted Zygmund Type Estimates for the Volterra Convolution Type

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Abstract:

Non-weight Zygmund-type estimates are obtained for mixed fractional integrals of the Volterra convolution type for a function of two variables defined by a mixed modulus of continuity.

Keywords: mixed fractional integral, mixed modulus of continuity, Zygmund type estimate, two variable functions, Volterra convolution type.

1. Introduction

An important stage in the study of fractional integro-differentiation of functions from generalized Hölder spaces is the obtaining of Zygmund-type estimates, i.e., an estimate of the modulus of continuity of a fractional integral through the modulus of continuity of the original function. A similar problem can be considered completely solved for the Hölder space of functions of one variable and power weights (see [1] see also [2]), as well as for the Hölder space of functions of two variables and power weights (see [4]-[13]). Obtaining an estimate of the Zygmund type for mixed fractional integrals with arbitrary kernels and has not been studied.

The main focus of the work is to obtain an estimate of the type of Zygmund majorizing a mixed modulus of continuity $\omega(\rho\tilde{k}\varphi; h, \eta)$ mixed fractional integral with a Volterra convolution type of integral weight constructions from a mixed modulus of continuity $\omega(\rho\varphi; h, \eta)$ its density $\varphi(x, y)$ with weight $\rho(x, y)$. These Zygmund type bounds and action theorems directly affect the nature of the improvement of the modulus of continuity by mixed fractional integration of the Volterra convolution type:

$$(\tilde{K}\varphi)(x, y) = \int_0^x \int_0^y k(x-t)k(y-s)\varphi(t, s)dt ds, \quad (1)$$

here we consider the degenerate kernels, as well as each $k(x)$ and $k(y)$ assumed to be close in some sense to a power function.

In this paper, we deal with arbitrary kernels, i.e., not necessarily power. We will consider the operator (1) in a rectangle $Q = \{(x, y): 0 < x < b, 0 < y < d\}$.

2. Preliminary

In this section, we present some well-known results and notation that we will need to present the issues under consideration (see [3]).

Everywhere in the results through, C, C_1, C_2, \dots we denote absolute constants that can have different meanings in different cases.

Definition 1. We say that $k(x) \in V_\lambda, \lambda > 0, k(x) > 0$, if $k(x) \in [0, l]$ and satisfies the conditions:

- 1) $k(x) \neq 0, x^\lambda k(x)$ - almost increases and $x^\lambda k(x)|_{x=0} = 0$;
- 2) $\exists \varepsilon > 0, 0 < \varepsilon < \lambda$, that $x^{\lambda-\varepsilon} k(x)$ - almost decreases;
- 3) $\exists C$, that

$$\left| \frac{k(x_1) - k(x_2)}{x_1 - x_2} \right| \leq C \frac{k(x^*)}{x^*}, \text{ if } x^* = \max(x_1, x_2). \quad (2)$$

Definition 2. Let given bounded on $[a, b]$ function $\varphi(x)$. Under modulus of continuity $\omega(x)$ understood the expression

$$\sup_{h \in [0, \delta]} |\varphi(x+h) - \varphi(x)| = \omega(\varphi; \delta), \quad 0 < \delta \leq b - a.$$

Definition 3. We denote by Φ^1 function class $\omega(\delta) \in (0, b-a]$ and satisfying the conditions

- 1) $\omega(\delta) > 0$ in $(0, b-a]$, $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$;
- 2) $\omega(\delta) \uparrow$ in $(0, b-a]$;
- 3) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$.

Below in the estimates, we need inequalities:

1) if $\omega(\varphi; h)$ is modulus continuity, then we have

$$x_2 \omega(\varphi; x_1) \leq C x_1 \omega(\varphi; x_2), \quad x_2 \leq x_1; \quad (3)$$

2) if $k(x) \in V_\lambda$, then $x_1^\lambda k(x_1) \leq C x_2^\lambda k(x_2)$ and $\exists 0 < \varepsilon < \lambda$,

$$x_2^{\lambda-\varepsilon} k(x_2) \leq C x_1^{\lambda-\varepsilon} k(x_1), \quad x_1 \leq x_2; \quad (4)$$

Lemma 1. Let $k(x) \in V_\lambda, \lambda > 0$ and $\omega(x) \geq 0$ almost increases, then for any $0 < x < l/2$, rightly

$$\omega(x)k(x) \leq C \int_x^l t^{-1} \omega(t)k(t)dt. \quad (5)$$

Consider a one-dimensional integral operator of the Volterra type convolutions

$$(K\varphi)(x) = \int_0^x k(x-t)\varphi(t)dt, \quad 0 < x < b. \quad (6)$$

Theorem 1. Let $k(x) \in V_\lambda, 0 < \lambda < 1$ и $\varphi(x) \in C([0, b]), \varphi(0) = 0$. Then for integral (4), the following estimate is valid

$$\omega(K\varphi, h) \leq C \left(hk(h)\omega(\varphi, h) + h \int_h^b \frac{k(t)\omega(\varphi, t)}{t} dt \right) \quad (7)$$

Proof. The integral (4) will be presented in the form

$$(K\varphi)(x) = \varphi(0) \int_0^x k(x-t) dt + \int_0^x k(x-t)[\varphi(t) - \varphi(0)] dt.$$

Since $\varphi(0) = 0$ then, we have $f(x) = \int_a^x g(t)k(t-a) dt$,

where $\varphi(t) - \varphi(0) = g(t)$.

Let $h > 0, x, x+h \in [0, b]$. Consider the difference

$$\begin{aligned} |f(x+h) - f(x)| &\leq C \left| \int_{-h}^0 k(h+t)[g(x-t) - g(x)] dt + \right. \\ &+ \left. \int_0^x (k(h+t) - k(t))[g(x-t) - g(x)] dt + g(x) \int_x^{x+h} k(t) dt \right| = \quad (8) \\ &= A_1 + A_2 + A_3. \end{aligned}$$

We estimate A_1 . Using inequality (3) and by definition 1, we have

$$\begin{aligned} A_1 &\leq \int_0^h \omega(\varphi, t)k(h-t) dt \leq C\omega(\varphi, h)k(h) \int_0^h \left(\frac{h}{h-t}\right)^\lambda dt \leq \quad (9) \\ &\leq Chk(h)\omega(\varphi, h). \end{aligned}$$

We estimate A_2 . Using inequality (2), we have

$$A_2 \leq C \int_0^x \frac{k(h+t)}{h+t} \omega(\varphi; t) dt.$$

Here we distinguish two cases: 1) $h \geq x$ and 2) $h < x$.

In the first case

$$\begin{aligned} A_2 &\leq Ch^{1+\lambda-\varepsilon} k(h) \int_0^x \frac{\omega(\varphi; t)}{(h+t)^{1+\lambda-\varepsilon}} dt = Chk(h) \int_0^{\frac{x}{h}} \frac{\omega(\varphi, ht) dt}{(1+t)^{1+\lambda-\varepsilon}} \leq \\ &\leq Chk(h) \int_0^1 \frac{\omega(\varphi, ht)}{(1+t)^{1+\lambda-\varepsilon}} dt \leq Chk(h)\omega(\varphi, h). \quad (10) \end{aligned}$$

In the second case A_2 , it can be represented as the

sum of two terms, i.e. $A_2 \leq \int_0^h + \int_h^{x-a} = A_2' + A_2''$, для

справедлива оценка (10). А для, имеем (for A_2' is rightly estimate (10). And for A_2'' , we have

$$A_2'' \leq Ch \int_h^x \omega(\varphi, t) \frac{k(t)}{t} dt \leq Ch \int_h^b \frac{\omega(\varphi, t)}{t} k(t) dt.$$

Finally, we estimate A_3 . By $x \leq h$:

$$A_3 \leq C\omega(\varphi, x)k(x+h)(x+h)^\lambda \int_x^{x+h} \frac{dt}{t^\lambda} \leq C\omega(\varphi, h)k(h)h.$$

If $x > h$, then using Lemma 1 and we have

$$A_3 \leq C\omega(\varphi, x)hk(x) \leq Ch \int_x^b \frac{k(t)}{t} \omega(\varphi; t) dt \leq$$

$$\leq Ch \int_x^b \frac{k(t)}{t} \omega(\varphi; t) dt.$$

Collecting all estimates for A_1, A_2, A_3 , we get (7).

Let a continuous function $\varphi(x, y)$ be defined in \mathbf{R}^2 .

We introduce the necessary notation

$$\left(\Delta_h \varphi \right) (x, y) = \varphi(x+h, y) - \varphi(x, y),$$

$$\left(\Delta_\eta \varphi \right) (x, y) = \varphi(x, y+\eta) - \varphi(x, y),$$

$$\begin{aligned} \left(\Delta_{h,\eta} \varphi \right) (x, y) &= \varphi(x+h, y+\eta) - \varphi(x, y+\eta) - \varphi(x+h, y) + \\ &+ \varphi(x, y), \end{aligned}$$

and

$$\begin{aligned} \varphi(x+h, y+\eta) &= \left(\Delta_{h,\eta} \varphi \right) (x, y) + \left(\Delta_\eta \varphi \right) (x, y) + \\ &+ \left(\Delta_h \varphi \right) (x, y) + \varphi(x, y). \quad (11) \end{aligned}$$

Now we introduce the following characteristics:

1) Private modules of continuity

$$\omega(\varphi; \delta, 0) = \sup_y \sup_{0 \leq h \leq \delta} \left| \left(\Delta_h \varphi \right) (x, y) \right| \text{ and}$$

$$\omega(\varphi; 0, \sigma) = \sup_x \sup_{0 \leq \eta \leq \sigma} \left| \left(\Delta_\eta \varphi \right) (x, y) \right|;$$

2) Mixed modulus continuity of order 1.1

$$\begin{aligned} \omega(\varphi; \delta, \sigma) &= \sup_{x, y} \sup_{\substack{0 \leq h \leq \delta \\ 0 \leq \eta \leq \sigma}} \left| \left(\Delta_{h,\eta} \varphi \right) (x, y) \right|, \text{ where} \\ &0 < \delta \leq b, 0 < \sigma \leq d. \quad (10) \end{aligned}$$

It follows from the definition $\omega(\varphi; \delta, \sigma)$ that this function belongs to each variable Φ^1 . In addition, we note that there is an inequality

$$\omega(\varphi; \delta, \sigma) \leq 2 \min \left\{ \omega(\varphi; \delta, 0), \omega(\varphi; 0, \sigma) \right\}. \quad (12)$$

3. Main result

In this section, we generalize Theorem 1 to the case $k(x, y) = k_1(x)k_2(y)$.

Theorem 2. Let $k_1(x), k_2(y) \in V_\lambda, 0 < \lambda < 1, \varphi(x, y) \in C(Q)$ and $\varphi(x, y)|_{x=0, y=0} = 0$. Then Zygmund type estimates are valid

$$\omega(\tilde{K}\varphi; h, 0) \leq C_1 \left[hk(h) \omega(\varphi; h, d) + h \int_h^b \frac{k(t)}{t} \omega(\varphi; t, d) dt \right], \quad (13)$$

$$\omega(\tilde{K}\varphi; 0, \eta) \leq C_2 \left[\eta k(\eta) \omega(\varphi; b, \eta) + \eta \int_\eta^d \frac{k(s)}{s} \omega(\varphi; b, s) ds \right], \quad (14)$$

$$\begin{aligned} \omega(\tilde{K}\varphi; h, \eta) \leq C_3 & \left[h\eta k(h)k(\eta) \omega(\varphi; h, \eta) + \right. \\ & + h\eta k(\eta) \int_h^b \frac{k(t)}{t} \omega(\varphi; t, \eta) dt + h k(h) \eta \int_\eta^d \frac{k(s)}{s} \omega(\varphi; h, s) ds + \\ & \left. + h\eta \int_h^b \int_\eta^d \frac{k(t)k(s)}{ts} \omega(\varphi; t, s) dt ds \right]. \end{aligned} \quad (15)$$

Proof. Using (11), (1) we will present in the form

$$\begin{aligned} (\tilde{K}\varphi)(x, y) &= \varphi(0,0) \int_0^x \int_0^y k(x-t)k(y-s) dt ds + \\ & + \int_0^x \int_0^y k(x-t)k(y-s) \left(\Delta_{t,s}^{1,1} \varphi \right) (0,0) dt ds + \\ & + \int_0^x \int_0^y k(x-t)k(y-s) [\varphi(t,0) - \varphi(0,0)] dt ds + \\ & + \int_0^x \int_0^y k(x-t)k(y-s) [\varphi(0,s) - \varphi(0,0)] dt ds. \end{aligned}$$

Since $\varphi(x, y)|_{x=0, y=0} = 0$ then we have

$$\begin{aligned} f(x, y) &= \int_0^x \int_0^y k(x-t)k(y-s) \left(\Delta_{t,s}^{1,1} \varphi \right) (0,0) dt ds = \\ &= \int_0^x \int_0^y g(t, s) k(x-t)k(y-s) dt ds. \end{aligned}$$

Let $h > 0, x, x+h \in [0, b]$. Consider the difference

$$\begin{aligned} [f(x+h, y) - f(x, y)] &= \\ &= \int_{-h}^0 \int_0^y k(h+t)k(s) [g(x-t, y-s) - g(x, y-s)] dt ds + \\ &+ \int_0^x \int_0^y [k(h+t) - k(t)] k(s) [g(x-t, y-s) - g(x, y-s)] dt ds + \\ &+ \left(\int_{-h}^x k(h+t) dt - \int_0^x k(t) dt \right) \int_0^y g(x, y-s) ds. \end{aligned} \quad (16)$$

Fair inequality

$$\begin{aligned} |f(x+h, y) - f(x, y)| &\leq C \left(\int_0^h \int_0^y k(h-t)k(s) \omega(\varphi; t, y-s) dt ds + \right. \\ &+ \int_0^x \int_0^y |k(h+t) - k(t)| k(s) \omega(\varphi; t, y-s) + \\ &+ \left. \int_x^{x+h} \int_0^y k(t)k(s) \omega(\varphi; t, y-s) dt ds \right) \leq \\ &\leq C_1 \left(\int_0^h k(h-t) \omega(\varphi; t, d) dt + \int_0^x |k(x+h) - k(t)| \omega(\varphi; t, d) dt + \right. \\ &\quad \left. + \int_x^{x+h} |k(x+h) - k(x)| \omega(\varphi; t, d) dt \right). \end{aligned}$$

Using the estimates for A_1, A_2, A_3 in the proof of Theorem 1, one can easily verify the inequality (13).

Having made a symmetric permutation (16), we can similarly get (14).

Let $h, \eta > 0, \forall x, x+h \in [0, b], \forall y, y+\eta \in [0, d]$. Consider the difference

$$\begin{aligned} \left(\Delta_{h,\eta}^{1,1} f(x, y) \right) &= \int_{-h}^0 \int_{-\eta}^0 \left(\Delta_{-t,-s}^{1,1} g \right) (x, y) k(t+h)k(s+\eta) dt ds + \\ &+ \int_0^x \int_0^y \left(\Delta_{-t,-s}^{1,1} g \right) (x, y) [k(h+t) - k(t)] [k(\eta+s) - k(s)] dt ds + \\ &\quad + g(x, y) \int_x^{x+h} \int_y^{y+\eta} k(t)k(s) dt ds + \\ &+ \int_{-h}^0 \int_0^y \left(\Delta_{-t,-s}^{1,1} g \right) (x, y) k(h+t) [k(\eta+s) - k(s)] dt ds + \\ &+ \int_0^x \int_{-\eta}^0 \left(\Delta_{-t,-s}^{1,1} g \right) (x, y) [k(h+t) - k(t)] k(s+\eta) dt ds + \\ &+ \int_x^{x+h} \int_{-\eta}^0 [g(x, y-s) - g(x, y)] k(t)k(\eta+s) dt ds + \\ &+ \int_{-h}^0 \int_y^{y+\eta} [g(x, y-s) - g(x, y)] k(t+h)k(s) dt ds + \\ &+ \int_x^{x+h} \int_0^y [g(x, y-s) - g(x, y)] k(t) [k(\eta+s) - k(s)] dt ds + \\ &+ \int_0^x \int_y^{y+\eta} [g(x-t, y) - g(x, y)] [k(t+h) - k(t)] k(s) dt ds. \end{aligned}$$

Fairness inequality

$$\begin{aligned} \left| \left(\Delta_{h,\eta}^{1,1} f \right) (x, y) \right| &\leq C \left\{ \int_0^h \int_0^\eta \omega(\varphi; t, s) k(h-t)k(\eta-s) dt ds + \right. \\ &+ \int_0^x \int_0^y \omega(\varphi; t, s) |k(h+t) - k(t)| |k(\eta+s) - k(s)| dt ds + \\ &+ \omega(\varphi; x, y) \int_x^{x+h} \int_y^{y+\eta} k(t)k(s) dt ds + \\ &+ \int_0^h \int_0^\eta \omega(\varphi; t, s) k(h-t) |k(\eta+s) - k(s)| dt ds + \\ &+ \int_0^x \int_0^\eta \omega(\varphi; t, s) |k(h+t) - k(t)| |k(\eta-s)| dt ds + \\ &+ \int_x^{x+h} \int_0^\eta \omega(\varphi; x, s) k(t)k(\eta-s) dt ds + \\ &+ \int_0^h \int_y^{y+\eta} \omega(\varphi; t, y) k(h-t)k(s) dt ds + \\ &+ \int_x^{x+h} \int_0^y \omega(\varphi; x, \tau) k(t) |k(\eta+s) - k(s)| dt ds + \\ &\quad \left. + \int_y^{y+\eta} \omega(\varphi; t, y) |k(t+h) - k(t)| |k(s)| dt ds \right\}. \end{aligned}$$

Estimating the obtained terms in the standard way, we arrive at inequality (15).

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